

Name: _____

Instructor: _____

Math 10560, Solutions Practice Exam 1.

- The Honor Code is in effect for this examination. All work is to be your own.
- No calculators.
- The exam lasts for 1 hour and 15 min.
- Be sure that your name is on every page in case pages become detached.
- Be sure that you have all 17 pages of the test.
- Each multiple choice question is worth 7 points. Your score will be the sum of the best 10 scores on the multiple choice questions plus your score on questions 13-16.

PLEASE MARK YOUR ANSWERS WITH AN X, not a circle!					
1.	(a)	(b)	(c)	(d)	(e)
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12.	(a)	(b)	(c)	(d)	(e)

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Multiple Choice _____

13. _____

14. _____

15. _____

Total _____

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Multiple Choice

1.(7 pts.) The function $f(x) = x^3 + 3x + e^{2x}$ is one-to-one. Compute $(f^{-1})'(1)$.

- $(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))}$
- By trial-and-error we determine that $f^{-1}(1) = 0$ (that is $f(0) = 1$).
- $f'(x) = 3x^2 + 3 + 2e^{2x}$.
- Hence $f'(f^{-1}(1)) = f'(0) = 5$.
- Therefore $(f^{-1})'(1) = \frac{1}{5}$.

(a) $\frac{1}{6+e}$

(b) $\frac{1}{6+2e}$

(c) 0

(d) $\frac{1}{4}$

(e) $\frac{1}{5}$

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2.(7 pts.) Solve the following equation for x :

$$\ln(x + 4) - \ln x = 1 .$$

- Amalgamating the logarithms, our equation becomes:

$$\ln\left(\frac{x+4}{x}\right) = 1.$$

- Applying the exponential to both sides, we get

$$\left(\frac{x+4}{x}\right) = e^1 = e$$

- Multiplying both sides by x , we get $x + 4 = ex$ and $x - ex = -4$.
- Therefore $x(1 - e) = -4$ and

$$x = \frac{-4}{1 - e} = \frac{4}{e - 1}.$$

(a) There is no solution.

(b) $x = \frac{4}{e - 1}$

(c) $x = \frac{4}{1 - e}$

(d) $x = e + 2$ and $x = e - 2$

(e) $x = \frac{4}{e - 1}$ and $x = \frac{4}{e + 1}$

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3.(7 pts.) Differentiate the function

$$f(x) = \frac{(x^2 - 1)^4}{\sqrt{x^2 + 1}}.$$

- Use logarithmic differentiation. (Take logarithm of both sides of equation, then do implicit differentiation.)

- $y = \frac{(x^2 - 1)^4}{\sqrt{x^2 + 1}}.$

- $\ln y = 4 \ln(x^2 - 1) - \frac{1}{2} \ln(x^2 + 1)$

- $\frac{1}{y} \frac{dy}{dx} = \frac{8x}{x^2 - 1} - \frac{x}{x^2 + 1}.$

- $f'(x) = \frac{dy}{dx} = \frac{x(x^2 - 1)^4}{\sqrt{x^2 + 1}} \left(\frac{8}{x^2 - 1} - \frac{1}{x^2 + 1} \right).$

(a) $f'(x) = \frac{x(x^2 - 1)^4}{\sqrt{x^2 + 1}} \left(\frac{8}{x^2 - 1} + \frac{1}{x^2 + 1} \right)$

(b) $f'(x) = \frac{(x^2 - 1)^4}{\sqrt{x^2 + 1}} \left(\frac{4}{x^2 - 1} - \frac{1}{x^2 + 1} \right)$

(c) $f'(x) = \frac{(x^2 - 1)^4}{\sqrt{x^2 + 1}} \left(\frac{4}{x^2 - 1} + \frac{1}{x^2 + 1} \right)$

(d) $f'(x) = \frac{(x^2 - 1)^4}{\sqrt{x^2 + 1}} \left(\frac{8}{x^2 - 1} - \frac{1}{x^2 + 1} \right)$

(e) $f'(x) = \frac{x(x^2 - 1)^4}{\sqrt{x^2 + 1}} \left(\frac{8}{x^2 - 1} - \frac{1}{x^2 + 1} \right)$

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4.(7 pts.) Compute the integral

$$\int_{2e}^{2e^2} \frac{1}{x \left(\ln \frac{x}{2} \right)^2} dx.$$

- Make the substitution $u = \ln \frac{x}{2}$ with $dx = xdu$. At $x = 2e$, have $u = 1$ and at $x = 2e^2$ have $u = 2$.

- $\int_{2e}^{2e^2} \frac{1}{x \left(\ln \frac{x}{2} \right)^2} dx = \int_1^2 \frac{1}{u^2} du = \left[-\frac{1}{u} \right]_1^2$

- $= -\frac{1}{2} + 1 = \frac{1}{2}.$

- (a) 2 (b) $\frac{3}{2}$ (c) $\frac{1}{2}$ (d) 1 (e) 0

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5.(7 pts.) Compute the limit

$$\lim_{x \rightarrow \infty} \frac{e^x - e^{-x}}{e^{2x} - e^{-2x}}.$$

Solution:

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{e^x - e^{-x}}{e^{2x} - e^{-2x}} &= \lim_{x \rightarrow \infty} \frac{e^x(1 - e^{-2x})}{e^{2x}(1 - e^{-4x})} \\ &= \lim_{x \rightarrow \infty} \frac{1 - e^{-2x}}{e^x(1 - e^{-4x})} = 0. \end{aligned}$$

(a) 1

(b) ∞

(c) e

(d) 0

(e) does not exist

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6.(7 pts.) Find $f'(x)$ if

$$f(x) = x^{\ln x}.$$

- One method is to use logarithmic differentiation. Let $y = f(x)$.
- $\ln y = \ln(x^{\ln x}) = (\ln x)(\ln x) = (\ln x)^2$.
- $\frac{y'}{y} = \frac{2 \ln x}{x}$.
- Therefore $f'(x) = y' = 2(\ln x)x^{(\ln x)-1}$.
- **Alternatively** we have $f(x) = (e^{\ln x})^{\ln x} = e^{(\ln x)^2}$
- $f'(x) = e^{(\ln x)^2} \frac{d(\ln x)^2}{dx} = e^{(\ln x)^2} (2 \ln x) \frac{1}{x} = \frac{x^{\ln x} 2 \ln x}{x} = x^{(\ln x)-1} 2 \ln x$.

- (a) $2(\ln x)x^{\ln x}$
- (b) $x^{\ln x} \ln x$
- (c) $2(\ln x)x^{(\ln x)-1}$
- (d) $x^{\ln x}(\ln x + 1)$
- (e) $x^{(\ln x)-1} \ln x$

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7.(7 pts.) Calculate the following integral.

$$\int_0^1 \frac{\arctan x}{1+x^2} dx .$$

- Make the substitution $u = \arctan x$ with $dx = (1+x^2)du$.

- $\int_0^1 \frac{\arctan x}{1+x^2} dx = \int_0^{\frac{\pi}{4}} u du$

- $= \left[\frac{u^2}{2} \right]_0^{\frac{\pi}{4}} = \frac{\pi^2}{32}.$

(a) $\frac{1}{2}$

(b) $\frac{\pi}{8}$

(c) $\frac{\pi^2}{32}$

(d) $\ln 2$

(e) $\frac{\pi^2}{8}$

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8.(7 pts.) If 100 grams of radioactive material with a half-life of two days are present at day zero, how many grams are left at day three?

- We have initial amount $m_0 = 100$ and half life $t_{1/2} = 2$ days.
- The amount left after t days is given by $m(t) = m_0 e^{kt} = 100e^{kt}$ for some constant k .
- To find the value of k , we use the fact that the half-life is 2 days. This tells us that $50 = 100e^{2k}$ or $\frac{1}{2} = e^{2k}$. Applying the natural logarithm to both sides, we get $\ln \frac{1}{2} = \ln e^{2k}$ or $-\ln 2 = 2k$.
- Therefore $k = \frac{-\ln 2}{2}$ and $m(t) = 100e^{-\frac{t \ln 2}{2}} = 100(e^{\ln 2})^{-\frac{t}{2}} = 100(2)^{-\frac{t}{2}}$
- After 3 days, we have $m(3) = 100(2)^{-\frac{3}{2}} = \frac{100}{\sqrt{2^3}} = \frac{100}{\sqrt{8}}$.

- (a) $\frac{100}{4^{1/3}}$ (b) $\frac{100}{2^{1/3}}$ (c) $\frac{100}{\sqrt{2}}$ (d) $\frac{100}{\sqrt{8}}$ (e) 50

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9.(7 pts.) $\lim_{x \rightarrow 0^+} (\cos x)^{\frac{1}{x^2}} =$

• This is an indeterminate form of type 1^∞ .

• We have

$$\lim_{x \rightarrow 0^+} (\cos x)^{1/x^2} = \lim_{x \rightarrow 0^+} \text{Exp} \left[\frac{\ln(\cos x)}{x^2} \right] = \text{Exp} \left[\lim_{x \rightarrow 0^+} \frac{\ln(\cos x)}{x^2} \right]$$

($\text{Exp}[x] = e^x$ here)

$$\bullet = \text{(by l'Hop)} \quad \text{Exp} \left[\lim_{x \rightarrow 0^+} \frac{\frac{1}{\cos x}(-\sin x)}{2x} \right] = \text{Exp} \left[\lim_{x \rightarrow 0^+} \frac{-\tan x}{2x} \right]$$

$$\bullet = \text{(by l'Hop)} \quad \text{Exp} \left[\lim_{x \rightarrow 0^+} \frac{-\sec^2 x}{2} \right] = e^{-1/2}$$

(a) e

(b) $e^{-\frac{1}{2}}$

(c) ∞

(d) 1

(e) Does not exist

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10.(7 pts.) The integral

$$\int_0^{\pi/2} x \cos(x) dx$$

is

- We use integration by parts with $u = x$, $dv = \cos x dx$. We get $du = dx$ and $v = \sin x$.
- Recall that $\int u dv = uv - \int v du$. Therefore $\int_0^{\pi/2} x \cos x dx = x \sin x \Big|_0^{\pi/2} - \int_0^{\pi/2} \sin x dx$
- $= \frac{\pi}{2} \sin \frac{\pi}{2} - 0 - [-\cos x]_0^{\pi/2} = \frac{\pi}{2} + [\cos \frac{\pi}{2} - \cos 0]$
- $= \frac{\pi}{2} + [0 - 1] = \frac{\pi}{2} - 1.$

(a) $\frac{\pi}{2} - 1$

(b) divergent

(c) 0

(d) $\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}$

(e) $1 - \frac{\pi}{2}$

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11.(7 pts.) Evaluate the integral

$$\int_0^{\pi/2} \sin^3(x) \cos^5(x) dx.$$

- Use the identity $1 - \cos^2(x) = \sin^2(x)$.
- Let $u = \cos(x)$, $du = -\sin(x)dx$
- $\int_0^{\pi/2} \sin^3(x) \cos^5(x) dx = \int_0^{\pi/2} (1 - \cos^2(x)) \sin(x) \cos^5(x) dx = - \int_1^0 (u^5 - u^7) du.$
- $= \int_0^1 (u^5 - u^7) du = \left[\frac{u^6}{6} - \frac{u^8}{8} \right]_0^1$
- $= \frac{1}{6} - \frac{1}{8} = \frac{1}{24}.$

(a) $\frac{1}{4}$

(b) $\frac{1}{24}$

(c) 0

(d) $-\frac{1}{24}$

(e) $\frac{\pi}{2}$

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12.(7 pts.) If you expand $\frac{1+x-x^2}{x(x+1)^2}$ as a partial fraction, which expression below would you get?

• $\frac{1+x-x^2}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$

- Multiplying the above equation by $x(x+1)^2$, we get

$$\begin{aligned} -x^2 + x + 1 &= A(x^2 + 2x + 1) + Bx(x+1) + Cx \\ &= Ax^2 + 2Ax + A + Bx^2 + Bx + Cx = (A+B)x^2 + (2A+B+C)x + A. \end{aligned}$$

- Comparing coefficients, we get $A = 1$, $2A + B + C = B + C + 2 = 1$, and $A + B = -1$. Therefore $B = -1 - A = -2$ and $C = -1 - B = 1$.

- The partial fractions decomposition of $\frac{2x+1}{x(x^2+1)}$ is therefore $\frac{1}{x} - \frac{2}{x+1} + \frac{1}{(x+1)^2}$.

(a) $\frac{1}{x} - \frac{2}{x+1} + \frac{1}{(x+1)^2}$

(b) $\frac{-2}{x} + \frac{1}{x+1}$

(c) $\frac{1}{x} + \frac{1}{(x+1)^2}$

(d) $\frac{1}{x} - \frac{3}{(x+1)^2}$

(e) $\frac{1}{x} + \frac{1}{x+1} + \frac{1}{(x+1)^2}$

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Partial Credit

You must show your work on the partial credit problems to receive credit!

13.(12 pts.)

Evaluate

$$\int \frac{x^2}{\sqrt{9-x^2}} dx.$$

Note: The formula sheet will help you with this problem.

Write your answer in terms of the original variable x and (if needed) replace all composite trigonometric functions (such as $\cos(\sin^{-1}(x/n))$ etc...) by algebraic combinations of x .

- Here we use the trigonometric substitution $x = 3 \sin \theta$, where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

- We have $x^2 = 9 \sin^2 \theta$, $dx = 3 \cos \theta d\theta$ and $\sqrt{9-x^2} = \sqrt{9-9 \sin^2 \theta} = 3|\cos \theta| = 3 \cos \theta$

- $\int \frac{x^2}{\sqrt{9-x^2}} dx = \int \frac{9 \sin^2 \theta}{3 \cos \theta} 3 \cos \theta d\theta = 9 \int \sin^2 \theta d\theta.$

- $= \frac{9}{2} \int (1 - \cos(2\theta)) d\theta = \frac{9}{2} \left[\theta - \frac{\sin(2\theta)}{2} \right] + C$

- We have $\theta = \sin^{-1} \frac{x}{3}$. Therefore $\int \frac{x^2}{\sqrt{9-x^2}} dx = \frac{9}{2} \left[\sin^{-1} \left(\frac{x}{3} \right) - \frac{2 \sin \theta \cos \theta}{2} \right] + C$

- Using a triangle, we get $\cos \theta = \frac{\sqrt{9-x^2}}{3}$ and

$$\begin{aligned} \int \frac{x^2}{\sqrt{9-x^2}} dx &= \frac{9}{2} \left[\sin^{-1} \left(\frac{x}{3} \right) - \frac{\frac{2}{9} x \sqrt{9-x^2}}{2} \right] + C \\ &= \frac{9}{2} \left[\sin^{-1} \left(\frac{x}{3} \right) - \frac{x \sqrt{9-x^2}}{9} \right] + C \end{aligned}$$

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14. (15 pts.) Let $C(t)$ be the concentration of a drug in the bloodstream. As the body eliminates the drug, $C(t)$ decreases at a rate that is proportional to the amount of the drug that is present at the time. Thus $C'(t) = kC(t)$, where k is a constant. The initial concentration of the drug is 4 mg/ml. After 5 hours, the concentration is 3 mg/ml.

(a) Give a formula for the concentration of the drug at time t .

(b) How much drug will there be in 10 hours?

(c) How long will it take for the concentration to drop to 0.5 mg/ml?

Solution: (a)

$$C(t) = C(0)e^{kt} = 4e^{kt}$$

$$C(5) = 3 = 4e^{k5} \quad (\text{solve for } k)$$

$$k = \frac{1}{5} \ln \left(\frac{3}{4} \right) \quad (\text{substitute into } C(t))$$

$$C(t) = 4 \left(\frac{3}{4} \right)^{\frac{1}{5}t}.$$

(b)

$$C(10) = 4 \left(\frac{3}{4} \right)^2 = \frac{9}{4}.$$

(c)

$$C(t) = 4 \left(\frac{3}{4} \right)^{\frac{1}{5}t} = \frac{1}{2} \quad (\text{solve for } t)$$

$$t = -5 \log_{3/4}(8) = \frac{-5 \ln 8}{\ln 3 - \ln 4}.$$

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15. (3 pts.) Please circle “TRUE” if you think the statement is true, and circle “FALSE” if you think the statement is False.

(a)(1 pt. No Partial credit) $\int \frac{1}{1+x^2} = \ln|1+x^2| + C.$

TRUE FALSE

(b)(1 pt. No Partial credit) $2^x = e^{x \ln(2)}.$

TRUE FALSE

(c)(1 pt. No Partial credit) $\ln\left(\frac{1}{x}\right) = -\ln(x)$ for all $x > 0.$

TRUE FALSE

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The following is the list of useful trigonometric formulas:

Note: $\sin^{-1} x$ and $\arcsin(x)$ are different names for the same function and $\tan^{-1} x$ and $\arctan(x)$ are different names for the same function.

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\sin 2x = 2 \sin x \cos x$$

$$\sin x \cos y = \frac{1}{2}(\sin(x - y) + \sin(x + y))$$

$$\sin x \sin y = \frac{1}{2}(\cos(x - y) - \cos(x + y))$$

$$\cos x \cos y = \frac{1}{2}(\cos(x - y) + \cos(x + y))$$

$$\int \sec \theta = \ln |\sec \theta + \tan \theta| + C$$

$$\int \csc \theta = \ln |\csc \theta - \cot \theta| + C$$

$$\csc \theta = \frac{1}{\sin \theta}, \quad \cot \theta = \frac{1}{\tan \theta}$$

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Please do NOT write in this box.

Multiple Choice _____

13. _____

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Total _____